Median Stable Distributions and Schröder's Equation

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Abstract: Median stable distributions were recently introduced as an extension of the standard idea of (mean) stable distributions. The extension defines stable distribution relative to an estimator. For mean stable distributions the mean's sampling distribution –recentered and rescaled–has the same distribution as the iid data. Median stable distributions replace the mean with the median in this definition. The sampling distribution of the median (instead of a sum) is a functional of the data's distribution so that its stable distribution is the solution to a functional equation. It came as a surprise to discover that the stable median functional equation is an instance of a famous functional equation due to Schröder from 1870. The renown of the equation is due to the way it incorporates iteration of functions, an important ingredient in what would become modern dynamical systems. The current paper reviews median stable distributions in light of the connection to Schröder's functional equation.

1 Introduction: Median Stable Distributions

Median stable distributions were introduced in a recent paper as a variation on (mean) stable distributions [1]. Such distributions are essentially those that make an estimator's sampling distribution the same as the distribution of the iid data on which the estimator is based.[1] Let $\hat{X} = \hat{X}(F)$ denote the sample mean based on iid data, $X_i(F) = X(F)$ whose common cdf is F. A mean stable distribution is a cdf H so that (after recentering by a μ and rescaling by a λ) the distribution of the estimator and the data are the same:

$$\lambda(\hat{X}(H) - \mu) \stackrel{d}{=} X(H) - \mu \tag{1}$$

Replacing the sample mean with the sample median in (1) defines a median stable distribution.

To fix ideas consider the simplest case of the median with $n = 3$ i.i.d observations. Let $\hat{X} = \hat{X}(F)$ denote the sample median, and let its (sampling) cdf be $M(x) = M(x :$

[1] Identifying a stable distribution in terms of its relation to an estimator is not standard. The usual presentation, motivated by the sample mean, is in terms of *sums* of i.i.d random variables; for example, [5]. Reinterpreting the standard definition in terms of the sample mean *estimator* has the advantage of generalization to new contexts.

$F) = \Pr[\hat{X} < x]$, which is given by,

$$M(x) = G(F(x)) \tag{2}$$

where the "G-function" is[2],

$$G(w) = 3w^2 - 2w^3, \quad w \in [0,1]. \tag{3}$$

(Note that $G(w)$ is the distribution of the sample median when the data is uniformly distributed on $[0,1]$).

The G-function is depicted in Figure 1 along with schematics illustrating how G turns F into M. The G-function is seen to be continuous, increasing, symmetric about $\frac{1}{2}$ with fixed points at $0, \frac{1}{2}, 1$. G is convex (concave) for $0 < w < \frac{1}{2}$ $(\frac{1}{2} < w < 1)$. $G(w) > w$ for $\frac{1}{2} < w < 1$, $G(w) < w$ for $0 < w < \frac{1}{2}$. These features hold for median G-functions for $n > 3$ as well as variations on the median considered below. While the analysis of classical mean stability is about sums of random variables, median stability concerns the behavior of functionals like G.

It is readily verified that the median of M and F are always the same. The sample median is median unbiased similar to the sample mean being mean unbiased. Without loss of generality the data is hereafter centered so that either its unique median is 0, or 0 is in the interval of medians, $[\mu^-, \mu^+]$.

A median stable distribution therefore will be a cdf H (depending on λ) satisfying (1), or, $H(x) = M[x : H(\lambda^{-1}x)]$, or substituting for M,

$$H(x) = G(H(\lambda^{-1}x)). \tag{4}$$

This defines the $n = 3$ version of median stable distributions.

Median stable distributions for $n = 2r+1$ are defined similarly: substitute G_r for G in (4) where G_r is the distribution of the sample median given iid data, uniform on $[0,1]$. This $G_r(w)$, $w \in [0,1]$ can be written as a transformation from the G in (3) (whose derivative is g) to;

$$G_r(w) = C_r \int_0^w g(t)^r dt. \tag{5}$$

where the constant C_r is determined so that $G(1) = 1$; $C_r = 6^{-r} \frac{(2r+1)!}{r!r!}$; see, e.g., [4]. The sampling cdf of the median given data with cdf F is, $G_r(F(x))$.

Schröder. I recently discovered that the defining functional equation for a median stable distribution is an instance of a famous functional equation considered by Schröder in 1870, [11]. As in our case, G and λ are the "givens" and the problem is to solve the functional equation (4) for the unknown H.

The fame of the equation is due its key role in what would eventually become the modern study of complex dynamical systems. This is due to the neat way that the equation involves iteration of functions.

[2] $\hat{X} < x$ if: (i) two-out-of-three, or (ii) three-out-of-three of the X_i are less than x. Two-out-of-three has probability, $F(x)^2(1-F(x))$, and can occur in three-choose-two equals 3 ways. Three-out-of-three can occur in 1 way, which has probability, $F(x)^3$. So, $M(x) = F(x)^3 + 3F(x)^2(1-F(x)) = 3F(x)^2 - 2F(x)^3 = G(F(x))$.

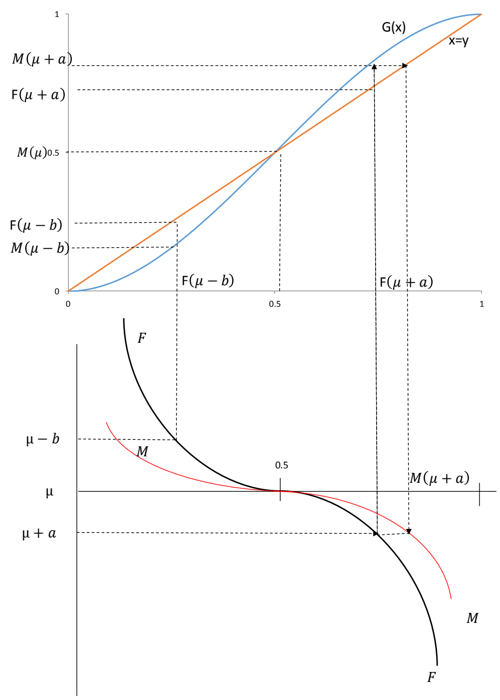

Figure 1: Mapping from F to M, via G

To illustrate, suppose H has an inverse (in our case, quantile) function H^{-1} so that (4) can be rewritten to express G in terms of H and H^{-1}:

$$G(w) = H(\lambda H^{-1}(w)). \tag{6}$$

The composition of $G(w)$ 2-times is,

$$G^{(2)}(w) = G(G(w)) = H(\lambda H^{-1}(H(\lambda H^{-1}(w)) = H(\lambda^2 H^{-1}(w)).$$

Continuing gives,

$$G^{(k)}(w) = H(\lambda^k H^{-1}(w)). \tag{7}$$

This k-composition works for the usual k, a positive integer. It also works for negative integers; $G^{(-k)}$ being the composition of G^{-1}, k-times. More interesting is that the RHS of (7) can be used to define functional composition for real, not necessarily, integer values of k. For example, $G^{(\sqrt{2})}(w) = H(\lambda^{\sqrt{2}} H^{-1}(w))$.[3]

[3]Also, $G^{(\sqrt{-1})}(w) = H(\lambda^{\sqrt{-1}} H^{-1}(w)))$.

Schröder was the first to use H-functions as a fundamental tool for understanding iteration of functions, a key feature of what would become today's dynamical systems, see [3]. For example, consider the sequence generated by a $G(w)$ process that starts at a $w_0 \in (0,1)$, where $w_{k+1} = G(w_k) = G^{(k)}(w_0)$. The associated H describes the evolution of the sequence, as well as the interpolated–continuous as a function of k– $w(k) = G^{(k)}(w_0)$.

Solving (4) for the unknown H in terms of G is not straightforward. In fact, "Schröder never succeeded in finding methods that guaranteed solutions.... He more or less admitted defeat, settling instead for what one might call the "Opposite Approach", wherein one begins with [H] and λ, and then defining [G] [via (6)], thereby obtaining ready made examples of "pre-solved" Schröder equations"([3], p.214).

It was not until 1884 that Koenigs [7] presented conditions such that an H-solution would exist for a given G. Typically there is no closed form solution, and H is expressed as the limit of a functional composition operation involving the known G function. (Solutions to the Schröder functional equation are sometimes referred to as Koenigs functions).[4]

Discussion of the connection between the Schröder Equation and median stable distributions is continued in the next section. Diagrams are used to describe relationships between G, H, and related quantities. Section 3 applies the results to the case where G corresponds to the median. As is typically the case, it is not easy to infer properties of H in terms of G, and a few features of median stable distributions from the previous paper are reviewed. One provides information about the density $h(x)$ near zero; the other concerns tail properties of H.

Section 4 deals with the remedian, an estimator defined recursively in terms of the regular median; see [10]. Many well known features of mean stable distributions including stable distributions as the domain of attraction for the limit of the estimator carry over to the remedian context.

Section 5 notes that median G_r-functions can be composed using different r-values, thus suggesting new varieties of estimators as well as their associated stable distributions. The final section takes up Schröder's "Opposite Approach" to derive G-functions corresponding to known H distributions. It also shows how an unknown H can be bounded in terms of a known distribution.

Discussion: Mean stable distributions do not depend on the number of observations. The normal or Cauchy is stable for any number of observations. n only determines the appropriate λ_n scaling factor; for the normal, $\lambda_n = \sqrt{n}$; for the Cauchy $\lambda_n = 1$.

The property of the mean that makes its associated stable distributions the same for any number of observations is its recursive property: the mean of means is the mean. For the regular median, however, the median of medians is not the median, and as a result median stable distributions are different for each n.

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[4]Many results are summarized in [13] and [8]. For an historical overview of the role of the functional equation in complex dynamics see, [6], and for the history of composition operators see, [12]. The usual setup in the literature has a fixed point of zero (rather than our fixed point of $1/2$). The conditions on the known function are typically for what would be our (convex) $G^{-1}(\frac{1}{2}+w)$. The original Koenig's conditions for G take the known function to be an analytic function around the fixed point; different conditions, for example, convexity, are noted in [13], p. 209-10.

The recursive property however holds for the remedian estimator and as a result mean stable features have analogues, but for the remedian. Like mean stable distributions, the stable distributions for the remedian are the same for all n, with only the scaling factor varying, similar to the mean stable situation. The domain of attraction feature also carries over. The limiting distribution of the remedian (which is a median stable distribution) given arbitrary, not necessarily stable, data will be a stable distribution. This is discussed in Section 4.

A final contrast between mean and median stable distributions concerns the scaling factor λ. For mean stable distributions there are λ-scaling factors less than one; with fat-tailed data the sample mean is actually more dispersed than the data so that it has to be "scaled-down" by a $\lambda < 1$. In contrast the sample median is always more concentrated than the data, so that its scaling factor is always greater than one.

2 Schröder's Equation

2.1 $\lambda = s$

Median G-functions such as the one depicted in Figure 1 are continuous, increasing, symmetric about $\frac{1}{2}$ with fixed points at $0, \frac{1}{2}, 1$. It is convex (concave) for $0 < w < \frac{1}{2}(\frac{1}{2} < w < 1)$, and $G(w) > w$ for $\frac{1}{2} < w < 1$, $G(w) < w$ for $0 < w < \frac{1}{2}$. The set of all such functions is denoted by $\mathscr{G}$. Given such a G, and a $\lambda > 1$, an H solution to (4) inherits features of G and will be: increasing, symmetric about 0 with $H(0) = \frac{1}{2}$, convex (concave) for $0 < w < 1/2$ $(1/2 < w < 1)$.

Further, an H solution for a given λ determines a scale family of distributions. (If $(H(x), \lambda)$ is a solution then so is, $H(\sigma x) = G(H(\lambda^{-1}\sigma x)$, which is a cdf for $\sigma > 0$.

In addition, an $(H(x), \lambda)$ for one λ determines solutions for all other $\lambda > 1$. If $(H(x), \lambda)$ solves (4) then so does $(H(x^{\alpha}), \lambda^{\alpha^{-1}})$, $\alpha > 0$. (Let $L(x) = H(x^{\alpha})$ so that, $L(x) = G(H(\lambda^{-1}x^{\alpha})) = G(H((\lambda^{-1/\alpha}x)^{\alpha})) = G(L(\lambda^{-1/\alpha}x))$, which says, $L(x), \lambda^{-1/\alpha}$ solves the functional equation).

Since a solution for any $\lambda > 1$ determines solutions for all λ we focus on the case, $\lambda = s$ where s denotes the slope of $G(w)$ at the fixed point, $w = \frac{1}{2}$. That is, the derivative of (4) is: $h(x) = g(H(\lambda^{-1}x)h(\lambda^{-1}x)\lambda^{-1}$, so $h(0) = g(\frac{1}{2})h(0)\lambda^{-1}$, so if, $0 < h(0) < \infty$, then $\lambda = g(\frac{1}{2}) = s$. The $\lambda = s$- particular case of the functional equation will therefore be:

$$H(x) = G(H(s^{-1}x)). \tag{8}$$

2.2 Schröder Solutions

Repeating the substitution of the LHS of (8) at $s^{-1}x$ into the RHS, gives, $H(x) = G^{(k)}(H(s^{-k}x)) = G^{(k)}(\frac{1}{2} + h(0)s^{-k}x) + o(s^{-k}x))$. Koenig [**?**] showed that

$$H(x) = \lim_{k\to\infty} G^{(k)}(\tfrac{1}{2} + s^{-k}x) \tag{9}$$

where $H(x)$ solves the function equation (8) with $h(0) = 1$.

Let $\mathscr{H}$ denote the set of all nondecreasing cdfs, symmetric about 0 with $h(0) = 1$. The $G \in \mathscr{G}$ (with $s = g(\frac{1}{2}) > 1$) associated with an $H \in \mathscr{H}$ is given by, $G(w) = H(sH^{-1}(w))$. Going the other way around, the $H \in \mathscr{H}$ associated with a $G \in \mathscr{G}$ is given by (9).[5]

2.3 $(\hat{X} \to G) \to H$

Figure 2.3 depicts connections between various quantities. The starting point is the left hand corner, a $G(w) \in \mathscr{G}$. The $\hat{X}$ to the left in parentheses indicates that this G might correspond to the sampling distribution of an estimator, such as the median, but at this stage $G \in \mathscr{G}$ without regard to whether it is a sampling distribution. The connection along the bottom, $G(w) \to H(x)$, is the mapping (9) with the indicated scaling factor $s = g(\frac{1}{2})$. This $H(x)$ is the stable distribution corresponding to G. The connection, $G(w) \to G^{(k)}(w)$, is the mapping to the new element of $\mathscr{G}$ via composition of G with itself, k-times. Similar to the inital G, the parentheses to the left indicates that this new element of $\mathscr{G}$ might correspond to the sampling distribution of an estimator. Finally, the connection $G^{(k)}(w) \to H(x)$ is the mapping (9) with the scaling factor s^k. A key feature of the diagram is that it shows the arrows connecting to the same H. This means that G and $G^{(k)}$, $k > 0$ have the same stable distribution (differing only in their scaling factors, s and s^k, respectively). [6]

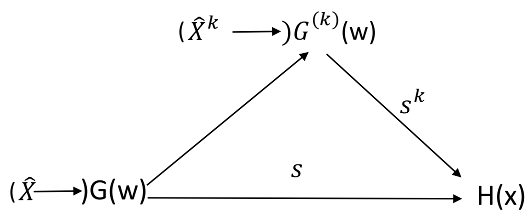

Figure 2: Schröder Diagram

3 Median G-functions

The corresponding diagram for the median is shown as Figure 3. The lower left hand corner now shows the median G function given 3 iid uniform observations, and to

[5]Analogous to the *standard* normal distribution, $H \in \mathscr{H}$ defines the *standard* cdf solution to the functional equation. The other solutions are obtained from the standard solution via $H(x : \sigma|s^{1/\alpha}) = H((\sigma x)^{\alpha})$, $\sigma > 0, \alpha > 0$.

[6]Notice that when G and $G^{(k)}$ correspond to the sampling distribution of an estimator, the fact their stable distributions are the same, is analogous to the mean stable result in which the stable distribution is the same for any number of observations (with the only the scaling changing from s to s^k). The normal is stable for $n = 3$ and $n = 9$ with the scaling factor going from $\sqrt{3}$ to $\sqrt{3}^2$; the Cauchy is stable for both values of n and the scaling goes from 1 to $1^2 = 1$.

the left, the associated median estimator, $\hat{X}$. The median given $2r+1$ observations is denoted by $\hat{X}_r$ and its associated $G \in \mathscr{G}$ is the mapping (5).

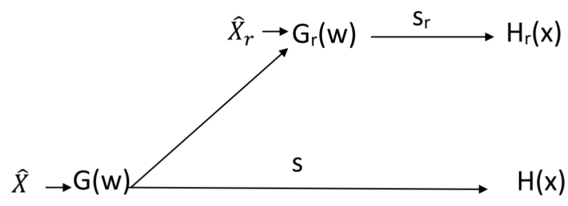

Figure 3: Median Diagram

As in the previous diagram the scaling factors are shown, along with the associated stable distributions. Unlike the previous diagram, $H(x) \neq H_r(x)$, the stable distributions are not the same, but depend on r.

3.1 $H_r(x)$: x near zero, $x \to \infty$

There is typically no closed form solution to the functional equation, and the solution (9) does not reveal what H looks like. In the previous paper two results were derived for $H(x)$ that reveal some information about what H looks like. The first shows that, while H is not normal, its log density at zero, is, like the normal proportional to x^2.

Theorem 1. $h_r(x) = \exp(-\alpha(r)x^2 h_r(0)^2 + o(x))$
where, $\alpha(r) = \frac{-4r}{s^2-1}$.

Proof. The derivative of (8) (suppressing the r-subscript) is $h(sx) = s^{-1}g(H(x))h(x)$, where $g(w) = s[1-4(w-1/2)^2]^r$, so: $\log(h(sx)) - \log(h(x)) = r\log(1-4(H(x)-1/2)^2)$. Dividing by x^2 and taking limits, $x \to 0$ gives, for the RHS:$-4rh(0)^2$, while the LHS is:

$$\lim_{x\to 0} \frac{s^2\log(h(x))}{x^2} - \frac{\log(h(sx))}{s^2x^2} = (s^2-1)\lim_{x\to 0}\frac{\log(h(x))}{x^2}.$$

So,

$$\lim_{x\to 0}\frac{\log(h(x))}{x^2} = -\alpha(r)h(0)^2,$$

where $\alpha(r) = \frac{-4r}{s^2-1}$. □

When $n=3$ ($r=1$), $\alpha(1) = 3.2$, and as $r \to \infty$ $\alpha(r)$ decreases to π, the coefficient for the associated normal density.

For the tails of H:

Theorem 2.

$$\lim_{x\to\infty} x^{-\alpha}\log H_r(-x) = c$$

where, $0 < c < \infty$, *and* α *satisfies,* $s^\alpha = r+1$.

Proof. The proof is based on the following result for the tail of $G_r(w)$. Suppressing the r-subscript, write $t(1-t) = \omega t$ where, for $0 < t < 1/2$, $1/2 < \omega < 1$; so

$$G(w) = C^{-1}\int_0^w [6t(1-t)]^r dt = C^{-1}(6\omega)^r \int_0^w t^r dt =$$

$$C^{-1}(6\omega)^r \tfrac{w^{(r+1)}}{(r+1)}.$$

The composition operator for G reduces to an operator on a monomial, $G^{(k)}(w) = (A(r)w^{r+1})^{(k)}$ where $A(r) = \frac{C^{-1}(6\omega)^r}{r+1}$. For monomials,

$$(bx^{\alpha})^{(k)} = b^{\frac{1-\alpha^k}{1-\alpha}} x^{\alpha^k}$$

so that,

$$\lim_{k\to\infty} \log \tfrac{(bx^{(\alpha+1)})^{(k)})}{\alpha+1} = \log x + \alpha^{-1}\log b$$

Hence for our problem,

$$\lim_{k\to\infty} \tfrac{\log G^{(k)}(w)}{(r+1)^k} = \lim_{k\to\infty} \tfrac{\log(A(r)w^{r+1})^{(k)}}{(r+1)^k} = \log x + r^{-1}\log A(r)$$

Finally, to prove the result, write $x = s^k x_0$, so

$$\lim_{x\to-\infty} x^{-\alpha}\log H(x) = \lim_{k\to\infty}(s^k w_0)^{-\alpha}\log H(-s^k x_0) = \lim_{k\to\infty} s^{-\alpha k} x_0^{-\alpha}\log G^{(k)}(w_0)$$

but, $s^{\alpha} = r+1$ so that

$$= \lim_{k\to\infty} \tfrac{x_0^{-\alpha}\log(A(r)w^{r+1})^{(k)}}{(r+1)^k} = x_0^{-\alpha}(\log x + r^{-1}\log A(r)) = c.$$

□

A final property for H follows from the known limiting distribution of the median: for $n\to\infty$, the ($\sqrt{n} = \sqrt{2r+1}$-scaled) median given iid uniform data is normal with mean zero and standard deviation $\frac{1}{2}$. In our notation,

$$= \lim_{r\to\infty} G_r(\tfrac{1}{2} + \frac{1}{\sqrt{2r+1}}x) = N(0:0,\tfrac{1}{2}).$$

In terms of our functional equation this means,

$$\lim_{r\to\infty} G_r(N(s_r^{-1}x:0,\tfrac{1}{2})) = N(x:0,\tfrac{1}{2})$$

(Verify noting

$$\lim_{n\to\infty} s_n^{-1}\sqrt{n} = \sqrt{\pi/2}$$

and $N((\pi/2)^{1/2}x:0,\sigma_M) = N(x:0,\frac{1}{2})$). Hence the normal distribution solves the functional equation as, $n = 2r+1\to\infty$.

4 Remedian G-functions

The remedian is the median of medians; see [10]. Its base-$(2r+1)$ version with $(2r+1)^k$ observations is defined recursively as the median of "data" that is itself medians, $(\hat{X}_{1\hat{X}_{(k-1)}}, \hat{X}_{2\hat{X}_{(k-1)}}, \ldots, \hat{X}_{(2r+1\hat{X}_{(k-1)}})$. The sampling distribution is, $RM_{r^k}(x:F) = G_r^{(k)}(F(x))$, [10]. (The composition of G_r with itself gives a new G function, which is the distribution of the remedian).

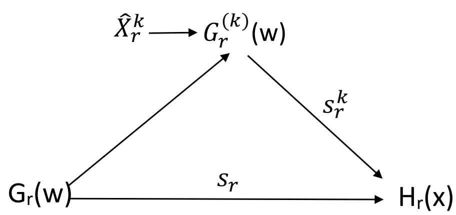

Figure 4: Remedian Diagram

Consider the scaled-by s_r^k distribution of the remedian; $G_r^{(k)}(F(s_r^{-k}x))$. If the data has the median stable distribution, $F = H_r$, then the cdf of the remedian is just $RM_{r^k}(x: H_r) = G_r^{(k)}(H_r(x)) = H_r(k)$. That is, the remedian given H_r data will also be have an H_r distribution. H_r is remedian stable for all $n = (2r+1)^k$, $k = 1,2,\ldots$.

4.1 Domain of Attraction

The stable distributions are the limiting distributions of the remedian when the data is stable, but they are also the limits given arbitrary, not necessarily stable, data.

Let the data have distribution, $F(x)$, where $f(x) > 0$ in a neighborhood of zero. The distribution of the remedian is, $G_r^{(k)}(F(s_r^{-k}x))$, or $G_r^{(k)}(\frac{1}{2} + f(\omega s_r^{-k}x)x)$, for $|\omega| < 1$, which as $k \to \infty$ is $H_r(f(0)w)$, the median stable distribution with scale, $1/f(0)$.

(More generally, let $F(x) = \frac{1}{2} + c(x)x^{\alpha} + o(x)$, where $c(x) > 0$ in a neighborhood of 0. Then, as in the above, the limiting distribution of the remedian is, $H_r(c(0)x^{\alpha})$).

5 Mixing G functions

The remedian was defined in [10] as the median of medians where the sub-medians all had the same "base"–number of observations. Since the composition of G functions is a G function, the remedian idea can be extended to a variety of new estimators by mixing r-median G functions.

For example, consider, $G_1(G_2(w))$. The corresponding estimator can be described as follows. Let $n = 15$ and put the data into an 5x3 array. The estimator defined as the median of the 5 row medians has distribution, $G_1(G_2(F(x)))$.

Consider, instead, the median of the 3 column medians. The distribution of this estimator is $G_2(G_1(F(w)))$ (which is not the same as the first estimator). While these two composite G functions are different, their derivatives at the $w = \frac{1}{2}$ fixed point are the equal, $(s_1 s_2 = s_2 s_1)$. Another–different–estimator with the same derivative at the fixed point is the randomized estimate that selects $G_1 G_2$ with probability p and $G_2 G_1$ otherwise. Which estimator is better? [7] Should you do the median of 3 then 5 or 5 then 3?

Each estimator has its associated stable distribution H_{12}, H_{21}. The functional equation for the first is, $H_{12}(x) = G_{12}(H_{12}(s_1 s_2)^{-1}(x))$, whereas for the second, $H_{21}(x) = G_{21}(H_{21}(s_1 s_2)^{-1}(x))$.

6 Pareto and Laplace

This section presents two examples in which H can be explicitly solved. The first is the simplest piecewise linear G that allows H to be computed by brute force. It leads to the Pareto distribution.

The second example uses Schröder's opposite approach that starts with a known H, the Laplace, and derives its associated G via, $G(w) = H(\lambda H^{-1}(w))$.

Finally, a given(G,H) pair can be used to infer properties for other G functions whose associated H features are difficult to discern. This is illustrated by showing that the median stable distribution (for which there is no closed form solution) is less dispersed than the Laplace.

The G functions for the examples are shown in Figure 6.

Pareto. Consider the polyhedral G function,

$$\begin{aligned} G(\tfrac{1}{2}+w) &= \tfrac{1}{2}+sw & 0 \le w < w_0 < \tfrac{1}{2} \\ &= \tfrac{1}{2}+sw_o + b(w-w_o) & w_o \le w \le \tfrac{1}{2} \end{aligned}$$

where $s > 1$, $b = \frac{1-2sw_o}{1-2w_o}$, and $G(\frac{1}{2}-w)$ is given by symmetry.

The associated $H(x)$ is a polyhedral–linear in-between the kinks–cdf whose value at the kink, $x_i = s^i w_o$, is:

$$H(s^i w_o) = G^{(i)}(\tfrac{1}{2}+w_o) = \tfrac{1}{2}+sw_o+(s-1)w_o\tfrac{b-b^i}{1-b}$$

(Verify using (9), and $H_k(x) = G^{(k)}(\frac{1}{2}+s^{-k}x)$, where $H_k(s^i w_o) = G^{(i)}(\frac{1}{2}+w_o) = \frac{1}{2}+sw_o+(s-1)w_o\frac{b-b^i}{1-b}$). Evaluating the density/slope of $H(x)$ gives

$$h(x) = (\frac{b}{s})^i$$

for, $s^i w_o < x < s^{i+1} w_o$, a linear interpolation of the Pareto distribution; that is, $h(x) = Ax^{-\beta}$, $\beta = 1 - \frac{\log b}{\log s}$.

Laplace. Let $H(x)$ be the standard ($h(0) = 1$) Laplace distribution and $H^{-1}(w)$ the associated inverse/quantile function:

[7] Answer: $G_2 G_1$ is better than $G_1 G_2$

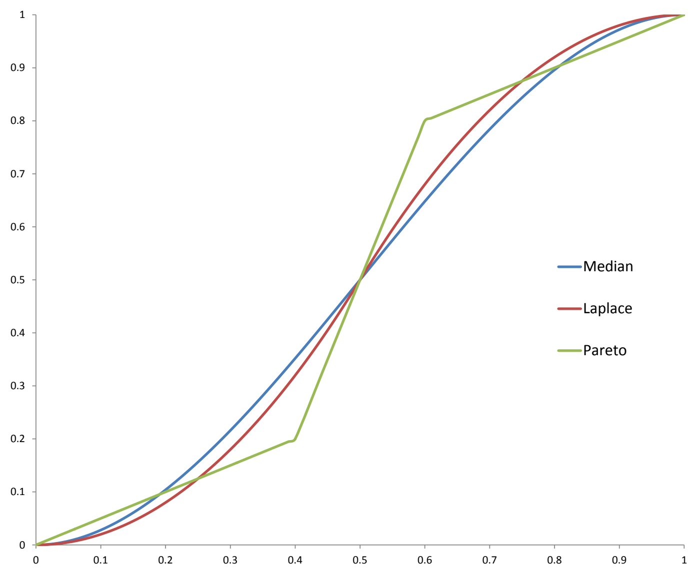

Figure 5: G functions

$$\begin{aligned} H(x) &= 1 - \tfrac{1}{2}e^{-2x} & x > 0 \\ H^{-1}(w) &= -\tfrac{1}{2}ln[2(1-w)] & w > \tfrac{1}{2} \end{aligned} \tag{10}$$

The distribution for $x < 0$ is given by symmetry, $H(x) = 1 - H(-x)$), as is the inverse for $w < \frac{1}{2}$; $H^{-1}(w) + H^{-1}(1-w) = 1$. The associated G-function is given using, $G(w) = H(\lambda H^{-1}(w))$, or:

$$\begin{aligned} G(w) &= 1 - \tfrac{1}{2}[2(1-w)^{\lambda}] & \tfrac{1}{2} < w < 1 \\ &= 2^{\lambda-1}w^{\lambda} & 0 < w < \tfrac{1}{2}. \end{aligned} \tag{11}$$

This pair, (G,H) can be used to create a new G associated with the same H, which has any prescribed $g(\frac{1}{2}) = \lambda$. This provides bounds for stable distributions that are otherwise difficult to discern.

To illustrate, let $G_1(w)$ denote the median G-function (2) with its $s_1 = g_1(\frac{1}{2}) = \frac{3}{2}$. We want to bound the unknown $H_1(x)$ relative to the Laplace.

Let $H_L(x)$ denote the Laplace distribution (10), and its G function (11), which is denoted by, $G_L(w|\lambda)$. Select $\lambda = s_1$, so that $G_1(w)$ and $G_L(w|s_1)$ have the same derivative at the fixed point: $g_1(\frac{1}{2}) = g_L(\frac{1}{2}|s_1) = s_1$.

The two G functions can be compared to infer a bound on the unknown $H_1(x)$ relative to the known $H_L(x)$. In particular, for $\frac{1}{2} < w < 1$, verify the inequality:

$$G_1(w) = 3w^2 - 2w^3 > 1 - \tfrac{1}{2}(1-w)^{\frac{3}{2}} = G_L(w|s_1)$$

which implies, $G_1^{(k)}(\frac{1}{2}+s_1^{-k}x) > G_L^{(k)}(\frac{1}{2}+s_1^{-k}x|s_1)$. Hence,

$$H_1(x) > H_L(x|s_1), \qquad x > 0 \tag{12}$$

While the stable distribution H_1 is not known precisely, (12) says it is less dispersed than the Laplace distribution.

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