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The St. Petersburg paradox and bounded utility

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I. *Bernoulli and Cramer*

In the 1738 *Papers of the Imperial Academy of Sciences in Petersburg,* Daniel Bernoulli originates the distinction between the mathematical and moral expectation of a lottery. The value of a lottery differs between persons not because one is thought luckier, expects his desires to be more closely fulfilled, but because the utility of an additional dollar differs for a poor man and a rich man. The value of a lottery is determined by its expected utility, utility depends on wealth, utility is a concave function of wealth, and, most specifically, the utility of additional wealth varies inversely with current wealth or $u(x) = \log(x)$. Bernoulli's paper, which had been presented to the Academy in 1731, contains a letter written by Gabriel Cramer (1728) which shows that Cramer independently arrived at the expected utility idea.

Bernoulli and Cramer's development of expected utility came about because both were seeking a solution to what is now called the St. Petersburg paradox. The paradox arises in a game where a fair coin is flipped until at the n-th flip heads first comes up and one then wins 2^n dollars. The realistic value of the game in actual life does not seem large even though the mathematical expectation is unbounded. Why? Bernoulli says it is because the value of the game is determined by expected utility, and he shows that the expected (log) utility value of the game is bounded and accords roughly with what persons would pay to play.

This much of the early history of expected utility and the St. Petersburg game is well-known. Also, well-known today is that Bernoulli's resolution of the paradox is unsatisfactory. As long as the utility function is unbounded, as $\log(x)$ is, the game can be always modified (by making prizes grow sufficiently fast) so that expected utility becomes unbounded while willingness to pay remains finite. (Let y_n, the prize when heads first comes up, grow so that $u(y_n) = 2^n$.) So, the paradox returns. To really avoid all paradox it is necessary to invoke both the expected utility principle and bounded utility (or else attribute the paradox to some other feature of the game).[1]

When was bounded utility first noticed to be essential to bound the value

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1. See Samuelson 1977, esp. 36–53, for other proposed solutions to the paradox.

of modified St. Petersburg games? Here is what Savage recounts in both the 1954 and 1972 editions of *The foundations of statistics:*

> None the less, as Cramer pointed out in his aforementioned letter [in Bernoulli's 1738 paper], the logarithm has a serious disadvantage; for, if the logarithm were the utility of wealth, the St. Petersburg paradox could be amended to produce a random act with an infinite expected utility (i.e., an infinite expected logarithm of income), that, again, no one would really prefer to the status quo [pp. 94–95].

That the mathematician Cramer would have noted this in his letter seems very believable; the need for bounded utility is obvious to anyone who knows about logarithms and the mathematical expectation. To those who learned the early history from Savage there is nothing remarkable about this and no reason to think that events did not occur as Savage presents them.[2] In fact, however, Cramer's letter does not say what Savage reports. There is no reference to the need for bounded utility in the letter reprinted in D. Bernoulli's paper. For some reason Savage made a mistake. I will present below some possible explanations for Savage thinking that the bounded utility fact was known by 1738.

To those who first learned the history from Savage the real surprise is not that Savage inadvertently gave too much credit to Cramer, but rather in finding out when the need for bounded utility was apparently first noted. The plausible-sounding 1738 date for first observing the need for bounded utility turns out to be wrong by two hundred years. The first person to make the bounded utility point was Karl Menger in 1934. Why would thus seemingly obvious fact go unrecognized for nearly two hundred years? Why did it take so long? We have a case where Savage's history, though wrong, seems much more plausible than the actual facts.

The next section presents some more historical facts and the last section briefly discusses some reasons for Savage's error and for the length of time it took to learn the necesssity of bounded utility. My facts are restricted to what is available in English and the explanations are restricted to obvious ones not requiring knowledge of background historical context regarding developments in probability and mathematics. We will see that none of the explanations is totally satisfactory.

II. *Historical Facts*

1738. Daniel Bernoulli's paper (in Latin) with the letter (in French) from Cramer to Nicolas Bernoulli is published. The paper will be translated to

2. Others too have taken this account to be believably true. Jeffrey 1965, 141, and 1983, 152, follow Savage except that he says that D. Bernoulli rather than Cramer was the first to note the necessity of bounded utility.

German in 1896 and English in 1954. The letter is presented by D. Bernoulli to show that Cramer independently arrived at the expected utility idea, though Cramer does not use the log(x) utility function. Cramer shows that the expected utility of the game is bounded under either of two utility functions. In the first he assumes $u(x) = x$ for $x<2^{24}$ and $u(x) = 2^{24}$ for $x>2^{24}$ and this bounds the expected utility at 13. This proposal would bound the value of modified games and it could perhaps be read in later years as saying that bounded utility was needed to avoid the paradox. But Cramer clearly did not intend or know of this interpretation. He never says that bounded utility is being used for this purpose and, more important, his first utility function is followed by one he likes better and which is subject to the same problem as log(x). He notes that $u(x) = \sqrt{x}$ brings the game's value down to 2.9 ducats which is "closer than is 13 to the vulgar explanation" (p. 35), but he then fails to notice that $\sqrt{x}\rightarrow\infty$ so that a modified game would still lead to unbounded expected utility.

1865. Todhunter's history is published. There are numerous references to the game and proposed resolutions of the paradox by many of those who helped develop probability theory up to the time of Laplace. No one notes the difficulty with the log(x) utility function. The history is filled with commentary by Todhunter pointing out both the false starts and the precursors of modern probability theory. If the problem with D. Bernoulli's resolution of the St. Petersburg paradox is obvious, it would seem that Todhunter would note it; but he does not.

1921. Keynes' book appears. On pp. 316–18 the game and its history are discussed; the history is taken from Todhunter 1865. After presenting the contents of Bernoulli's paper he says (p. 318):

> As a solution of the Petersburg problem this line of thought [by Cramer] is only partially successful; if increases of "physical" fortune beyond a certain finite limit can be regarded as "morally" negligible, Peter's claim for an infinite initial stake from Paul is, it is true, no longer equitable, but with any reasonable law of diminution for successive increments Paul's stake will still remain paradoxically large.

Keynes comes very close here but does not quite realize that bounded utility is necessary. He does not demonstrate how modified games create unbounded expected utility in case $u(x)\rightarrow\infty$ and he finishes the above with a note regarding Bernoulli being the first to use the diminishing marginal utility of money hypothesis.

1934. Menger's paper is published. It contains the first statement of the need for bounded utility. The paper is in German; it will be translated into English in 1967. Menger is aware of both the problem with log(x) and the fact that no one seems to have noted that there is a problem. After presenting Bernoulli's solution he says:

> We begin by demonstrating the solution of the Petersburg paradox according to the logarithmic formula for subjective value (contrary to the views expressed for more than one hundred years in the textbooks on probability theory) is unsatisfactory on formal grounds [p. 217].

He goes on to explain how modified games recreate the paradox unless $u(x)$ is bounded. The paper also reviews other proposed solutions to the paradox.

The 1940s. References to Menger's paper begin to appear in English. The earliest such references I have found are two by Tintner (1941 and 1942). Tintner, however, does not mention Menger's point about bounded utility. (He references Menger's suggestion that moments other than just the mean of the utility distribution might influence risky choice.) Von Neumann & Morgenstern 1944, 28, refer to Menger's paper, but they too do not mention the need for a bounded utility. The earliest reference I have found which has both a reference to Menger *and* a statement of Menger's bounded utility result is Carnap 1950, 272–73.[3]

Early 1950s. By the early 1950s it seems that many people recognize the need for bounded utility. Some have read and note Menger's contribution, and some seem to know the need for bounded utility even if they have not read Menger. We also begin to see how some of the secondary references to Cramer's letter might lead Savage to think that the need for bounded utility originated in 1738. Here are some examples.

I. J. Good 1950, 54, states the usual version of the game, notes Bernoulli's solution via his reading of Todhunter 1865, and then says this is inadequate; there has to be an upper bound, because we can create new paradoxes with modified games by allowing rapidly growing prizes. There is no reference in the book to Menger's work.

Stigler 1950, 373–75, presents the history of the game via Todhunter 1865, and he uses the German translation as his source for the Bernoulli 1738 paper. In a footnote Stigler notes the need for an upper bound on utility. There is no reference to Menger. This paper and Good's show that boundedness is now recognized and thought to be obvious and known for a long time.

Arrow 1951, 421, tells about the Menger paper and he gives credit to Menger for first noticing the need for bounded utility. After presenting Menger's argument Arrow has a footnote which may have been the cause of Savage's later mistake. It reads: "The resolution of the St. Petersburg paradox by means of a bounded utility function was first proposed by the eighteenth-century mathematician Cramer in a letter printed in D. Ber-

3. Savage reviewed Carnap's book (Savage 1952) and the book is referenced in Savage 1954, but not in the section on history. When Savage comes to write his historical comments I believe he has forgotten about the Menger comments in Carnap. Savage is instead using as secondary references some of the articles discussed next.

noulli's original paper" (p. 421). It is very easy to see how this could be incorrectly read to say that Cramer was the one who noted the log(x) problem and the *necessity* of bounded utility.

In a letter commenting on a previous version of this paper, Arrow says that he remembers learning of the Menger paper either from Oskar Morgenstern or perhaps from Menger himself. That Morgenstern knew of Menger's paper and used it to persuade Von Neumann to undertake a formal treatment of utility in the second and third editions of *Theory of games and economic behavior* is noted by Menger in the 1967 introductory note to his paper. Arrow may have heard about the paper from Menger himself because at the time that Arrow was preparing his 1951 paper, Menger was at the Illinois Institute of Technology in the mathematics department. Arrow remembers that Menger sometimes attended the Cowles Commission Seminars at the University of Chicago in the early 1950s and it could have been there that he first heard of Menger's paper. (Menger stayed at IIT until his death in 1985 at the age of 87.)

1954. At the beginning of the short section "Historical and critical comments on utility," (p. 91) Savage takes note of the above papers by Stigler (1950) and Arrow (1951) so he probably knows of the need for bounded utility and he possibly knows of Menger. But there is no reference to Menger's paper in either edition of *Foundations of statistics*. It is tempting to think that Savage's mistake occurred because he merely forgot how he knew about the log(x) problem. The only problem with this explanation is that Savage presents a very detailed review of Bernoulli's paper based on the German edition; see pp. 92–95. Savage covers virtually every point and in the same sequence as those in Bernoulli (1738). To me it appears that Savage is writing with Bernoulli's paper right in front of him. But then at the end Savage attributes too much to Cramer. It would seem that the mistake went unrecognized, because it is still in the 1972 edition.

1977. Samuelson's extensive review article on the St. Petersburg game is published. He says (p. 32), "After 1738, nothing earthshaking was added to the finds of D. Bernoulli and his contemporaries until the quantum jump in analysis provided by Karl Menger (1934)." Having learned the history of the problem from Savage's account, I originally thought Samuelson had made a two-hundred-year mistake. Instead he does have the history correct; he uses Todhunter 1965 and Arrow 1951. However, Samuelson does not note the discrepancy between his and Savage's account, and he merely reports the two-hundred-year gap between D. Bernoulli and Menger as if it were not that remarkable.

III. *Explanations*

My current guess regarding Savage's mistake is that it arose because Savage knew that bounded utility was needed, he knew Cramer's letter mentioned a bounded utility function, and as he looked away from Bernoulli's paper the two thoughts intermingled and he ended up giving Cramer credit for more than what was in his letter. The mistake must have gone unrecognized because it is still in the 1972 edition of *Foundations*.

An explanation for the two-hundred-year gap is more difficult. To see the need for bounded utility it is essential to think of modified St. Petersburg games with altered and rapidly growing prizes. Perhaps no one was accustomed to thinking of modifying games in this way. This explanation is wrong. In fact, as Todhunter 1865, 133, reports, the regular version of the game grew out of games which were successively modified by increasing the growth rate of prizes. In the precursor of the game "n" is the toss when a six first shows up in the case of a die, and we ask first for the value of the game if the prize is n, then if the prize is 2^n or n^2 or n^3. Again, these questions are arising prior to the St. Petersburg game. If one is thinking along these lines, it would seem natural to ask for the value of the St. Petersburg game with prizes growing like 2^{2^n}. But apparently no one did.

Another possibility for the gap is a problem-shift explanation. This says that many would have noticed the need for bounded utility if they had just thought about it. No one noticed, because they were focused on other kinds of problems. The problem with this is that practically every important figure in the development of probability from D. Bernoulli onwards has had something to say about the St. Petersburg problem; for examples, see the various Petersburg references in Todhunter 1865. Furthermore, those who favored alternative explanations for the paradox (for example, the problem is with the unbounded possible prizes which make the game impossible to actually play, or the small probabilities in the tail sequence of the game are evaluated as zero), could make a point against expected utility by noting the inadequacy of Bernoulli's solution. But no one made such a point.

The only other explanation I have is that there really is no explanation. The need for bounded utility seems an intrinsically obvious fact and those who, following Savage, thought this was noted in 1738 or thereabouts are amazed to find out that 1934 is the correct date. In going through the old probability and statistics texts I kept expecting to find some pre-1934 reference to the need for bounded utility. My intent was not to diminish Menger's work; the 1934 paper is certainly the first to note the need for bounded utility and to be (eventually) widely read. It is just that the problem with the $\log(x)$ solution to the St. Petersburg problem seems so obvious that I wondered then and still continue to wonder how the eminent mathematicians who analyzed the game should have failed to notice it.

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