Properties of the Cross-Sectional Distribution of Daily Equity Returns

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Abstract

We present statistical properties of the daily cross sectional distribution of equity returns for the 1000 largest capitalization stocks in the CRSP database during the period 1991 to 2002. The investigation is motivated by the very large changes in the distribution that occurred between 1998 and 2001. Summary statistics for the dispersion, skewness, and tail length are presented. A skew t-distribution is used to model the distribution and daily MLE estimates for the parameters are presented. We find the change during 1998 to 2001 was concentrated in the scale parameter with little change in the tail length. In addition the cross sectional distribution shares the fat-tailed property of time series variation of prices with an estimated degrees of freedom parameter of about 3.4.

Key Words: cross-sectional return distribution, quantile skewness, tail-length, generalized skew t-distribution

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1 Introduction

The capital asset pricing model’s focus on expected returns has led to investigations of the cross sectional distribution of expected equity returns. As is well known, for example, Fama and French (1993, 1996) examined expected returns as a function of a variety of characteristics such as size, leverage, past returns, dividend-yield, earnings-to-price ratios and book-to-market ratios. Further, Jegadeesh and Titman (1993) looked at variations in expected returns as a function of size and book-to-market, while Daniel and Titman (1997) considered variations in expected returns as a function of firm characteristics.

We extend previous analysis and consider variations in the entire distribution of equity returns. It is noted that stock indices only reflect the central location (mean) of the entire ensemble of stocks in the market. Although it provides important information of the market condition reported daily to the public, a single index value is unable to describe the overall picture of the market. We show that, the cross sectional variance, skewness, and actually the entire cross-sectional distribution provide a more complete and accurate picture of the market. Another reason for interest in the distribution is its impact on the potential profitability of long-short equity strategies. When dispersion is large, buying stocks with returns in the right-hand tail and shorting stocks with returns in the left-hand tail produces greater profits than when dispersion is small. The cross-sectional distribution also affects risk. At one extreme, risk is slight when the cross-sectional distribution is concentrated around its mean or median value. At the other extreme portfolio risk is larger when there is a large dispersion in the cross-sectional distribution, especially for portfolios that concentrate on a small number of stocks or sectors.

Lillo and Mantegna (2000a) examined the ensemble of daily stock returns. They considered the first four central moments and characterized the statistical properties of those moments by investigating their probability density function. Lillo and Mantegna (2000b) extended their analysis and found that properties of the return distribution change over time. Campbell et al. (2001) consider the variations in the volatility of individual stocks and their
relation to market and industry volatility. This is similar to the study by Hamilton and Lin (1996) and Schwert (1989a, 1989b) who find that economic recessions are the single largest factor affecting the volatility of individual stocks.

We go beyond the analysis of individual stock or market volatility and consider the changing distribution of cross-sectional equity returns. After remaining constant for many years, the distribution began to change in mid-1998 so that formerly unlikely tail events became much more likely. Dispersion grew very quickly until early 2000 and since then has gradually returned to pre-1998 levels. In this paper we report the changing properties of the equal-weighted cross-sectional distribution of the 1000 largest capitalization stocks between 1991 and 2002.

The paper is organized as follows. Section 2 presents summary measures for the daily cross section of returns. These include the daily dispersion, skewness, and tail-length. In section 3 we introduce a skew t-distribution as a model for the cross-sectional distribution. Section 4 presents MLE estimates of the parameters of the distribution for each day. We corroborate the changing dispersion indicated in section 2, while also presenting estimates for the skewness and tail length of the cross-sectional distribution.

2 Characteristics of the Cross-sectional Return

The data in our study is from the CRSP data base. The time index $t$ ranges over the 3027 days between 1991 and 2002. Let $r_{it}$ denote the (realized) return of the $i$th stock in the stock universe $U_t$ at time $t$.

The empirical cross-sectional distribution (CDF) of the 1000 largest stocks is,

$$
\tilde{F}_t(z) = \frac{1}{1000} \sum_{i \in U_t} I(r_{it} < z),
$$

where $I(\cdot)$ is the indicator function. The associated quantile (inverse) function is denoted by $Q_t(\theta)$. We start the investigation by examining the moments of the cross-sectional return. We will also investigate robust measures of the distribution.
Central Tendency

The classical measure of central tendency is sample mean. For example, the market indices are weighted sample mean over an ensemble of stocks. The daily cross sectional mean is shown in Figure 1(a). A robust measure of the central tendency is the median, \(Q(0.5)\), which is shown in Figure 1(b). The mean and median series have similar pattern over 12 years.

Dispersion

We utilize several measures of dispersion. The first is the daily variance of the cross section, \(V(R_t)\); \(R_t\) is the random variable whose CDF is \(\tilde{F}_t\). The daily variance is depicted in Figure 1(c). It is seen that the variance remained relatively constant until 1998, but then increased until 2001. After that, the variance returns to pre-1998 levels. Quantile differences provide an alternative measure of dispersion that, unlike the variance, are not sensitive to extreme outliers. We use \(Q(.90) - Q(.10)\), which is presented in Figure 1(d). It shows a pattern similar to the variance picture.

Skewness

The classical skewness measure is based on the third central moment. Suppose \(\bar{x}\) is the sample mean of \(x\) and \(s\) is the sample standard deviation. The sample skewness is defined as:

\[
\frac{1}{N} \sum_{j=1}^{N} \left( \frac{x_j - \bar{x}}{s} \right)^3.
\]

We plot the daily skewness in Figure 1(e). The skewness measures vary around 0, but tend to be negative after 2000.

Hinkley (1975) suggested the following alternative class of skewness measures

\[
\frac{(Q(1 - p) - Q(0.5)) - (Q(0.5) - Q(p))}{Q(1 - p) - Q(p)}.
\] (1)

Note that this measure is scale invariant. Further the coefficient is 0 for a symmetric distribution, greater than 0 for a distribution skewed to right and less than 0 for a distribution skew to left. We present this skewness measure for \(p = .05\). Over the entire 12 year period, it averages 0.048 with a standard deviation 0.17. Figure 1(f) shows the daily skewness. It is
seen that the measure increases after 1998, which coincides with the volatility trend.

**Tail-length**

Financial time series are well-known for having fat tails, (e.g. see Tsay (2002)). The traditional measure of tail-fatness or peakedness is kurtosis. Suppose \( \bar{x} \) is the mean of \( x \) and \( s \) is the standard deviation. The sample excess kurtosis is defined as:

\[
\frac{1}{N} \sum_{j=1}^{N} \left( \frac{x_j - \bar{x}}{s} \right)^4 - 3.
\]

The daily excess kurtosis measure is plotted in Figure 1(g). Over the entire period, the excess kurtosis measures are all greater than 0.

Darlington (1970) showed however that the standard kurtosis measure can instead reflect unimodality vs bimodality. Hence we define the robust tail-length index

\[
\frac{Q(0.95) - Q(0.05)}{Q(0.75) - Q(0.25)}.
\]

This index is scale invariant. The tail length values are presented in Figure 1(h). For comparison purposes, we also indicate the corresponding tail length measures for the normal and various t-distributions. From lowest to highest, the reference lines are the tail length indices computed from standard normal (2.44), student t-distribution with degrees of freedom 4 (2.88), 3 (3.08) and 2 (3.58). The figure indicates that the cross sectional distribution tends to be fatter tailed than the normal distribution. A t-distribution with degrees of freedom about 3 or 4 fits the majority of the days.\(^1\)

### 3 Skew T-distribution

The evidence above suggests that daily observations follow a skew, fat-tail distribution. McDonald and Newey (1988) introduced the generalized t-distribution and Theodossiou (1998) developed its skewed extension to accommodate the skewness and excess

\(^1\)Note however that the tail length index is not invariant to skewness. For example, a symmetric student t distribution with degree of freedom 3 has tail length 3.08, while a corresponding asymmetric student t distribution with skewness parameter 0.5 has a tail length index of 2.93.
Figure 1: Classical and robust measurements of the first four moments
(a): daily mean; (b): daily median (Q(0.5)); (c): daily variance; (d): daily quantile difference between quantile 0.9 and quantile 0.1; (e): skewness measure based on the third moment; (f): quantile skewness with p = 0.05; (g): excess kurtosis based on fourth moment; (h): tail-length index with reference lines over 12 years.
kurtosis present in financial data. Jones and Faddy (2003) also proposed a tractable skew t-distribution and developed likelihood inference for its parameters. Based on the facts presented in section 2 we develop a skew t-distribution with location and scale parameter as a model for cross-sectional returns. Fernandez and Steel (1998) presented a general method for transforming a symmetric distribution into a skewed distribution. We follow their approach to construct our skew t-distribution. Different from the previous versions, the density function we propose is easy to interpret since it is similar to the form of the student t-distribution and each parameter has an intuitive interpretation. Further, the cumulative distribution, the quantile function, the moments and other properties of the proposed skew t-distribution can be obtained without complicated computations.

3.1 Definition and Properties

We consider a density function that consists of two separated pieces, but retains unimodality and continuity. It controls for the skewness with a single parameter. Specifically, the skew t-distribution has density

$$g(x|\mu, \sigma, \nu, \lambda) = \begin{cases} \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\pi\nu\sigma^2}} (1 + \frac{(x-\mu)^2}{(1-\lambda)^2\sigma^2\nu})^{-\frac{\nu+1}{2}} & x < \mu \\ \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\pi\nu\sigma^2}} (1 + \frac{(x-\mu)^2}{(1+\lambda)^2\sigma^2\nu})^{-\frac{\nu+1}{2}} & x \geq \mu \end{cases},$$

where $\mu$ is location parameter, $(-\infty < \mu < \infty)$, $\sigma$ is scale parameter, $(\sigma > 0)$, $\lambda$ is skewness parameter, $(-1 < \lambda < 1)$, and $\nu$ is degrees of freedom, $(\nu > 0)$. The density is continuous and smooth at $\mu$. The standard normal distribution, student t-distribution, skew normal distribution are all special cases of the proposed skew t-distribution. In Figure 2, we present the PDF of a skew t distribution with $\mu = 0$, $\sigma = 0.8$, $\lambda = 0.7$, and $\nu = 3$ as well as the standard normal distribution and the student t-distribution with degrees of freedom 3.

Clearly, $g(x|\mu, \sigma, \nu, \lambda)$ retains the unique mode at $\mu$ but loses symmetry when $\lambda \neq 0$. In fact, when $\lambda = 0$ we deduce $g(x|\mu, \sigma, \nu, 0) = f(x|\mu, \sigma, \nu)$, where $f$ is the density function of a symmetric t-distribution with location and scale parameter $(\mu, \sigma)$ and degrees of freedom $\nu$. 7
We see that $\lambda$ controls the allocation of mass to each side of the mode,

$$P(x \geq \mu|\mu, \sigma, \nu, \lambda) = \frac{1 + \lambda}{2}, \quad \text{and} \quad P(x < \mu|\mu, \sigma, \nu, \lambda) = \frac{1 - \lambda}{2}.$$

The way in which the skewness parameter $\lambda$ intervenes in the density function implies that $g(x - \mu|\mu, \sigma, \nu, \lambda) = g(\mu - x|\mu, \sigma, \nu, -\lambda)$ so that changing the sign of $\lambda$ produces a mirror image of the density function around $\mu$.

The $r$th order moment of our skew t-distribution exists and is finite if and only if all the $i$th order moments ($i \leq r$) of the corresponding symmetric student t-distribution $p(\cdot)$ with the same parameters exist and are finite. In particular, we obtain

$$E(x^r|\mu, \sigma, \nu, \lambda) = \sum_{i=0}^{r} \binom{r}{i} \frac{\mu^{r-i} \sigma^i [(1 + \lambda)^{i+1} + (-1)^i (1 - \lambda)^{i+1}] M_i}{r}$$

where $M_i = \int_0^\infty s^i p(s) ds$. Some of the properties of generalized skewed univariate distribution obtained by Fernandez and Steel (1998) also apply to this proposed skew t-distribution. Specifically $E(x^r|\mu, \sigma, \nu, \lambda)$ is real-valued only for integer $r$. The finite moments exist only
for degrees of freedom \( \nu > r \). The unimodality of \( p(\cdot) \) implies that \( M_1 = \infty \) for \( r \leq -1 \). Thus, we concentrate on positive integer order moments. Given the degree of freedom \( \nu \), location \( \mu \) and scale parameter \( \sigma \), the \( r \)th order moment is an \( r \)th order polynomial function of the skewness parameter \( \lambda \). Consequently, \( \min \lambda E(x^r|\lambda) = E(x^r|\lambda = 0) \) for all \( r \). Expressions for standard student t’s moments are readily available from \( M_r = \int_0^\infty s^r p(s) ds \).

The classical skewness measure based on the third moment is available from the moment formula. An alternative skewness measure provided by Arnold and Groeneveld (1955) is \( 1 - 2F(\text{mode}) \). From the expression of \( g \) in (3), we see that the mode is reached at \( \mu \). Hence

\[
1 - 2F(\text{mode}) = 1 - 2F(\mu) = 1 - 2 \left( \frac{1 - \lambda}{2} \right) = \lambda,
\]

so the skewness measure of Arnold and Groeneveld (1955) is exactly equal to the skewness parameter in the distribution.

The proposed skew t-distribution density has tails that behave as \( |x|^{-(\nu+1)} \) as \( x \to \pm \infty \). Both tails match those of the symmetric t-distribution with degrees of freedom \( \nu \).

### 3.2 Estimators for the Parameters of the Skew T-distribution

The log likelihood of skew t-distribution is

\[
\log L = N \left( \log \Gamma \left( \frac{\nu + 1}{2} \right) - \log \Gamma \left( \frac{\nu}{2} \right) - \frac{1}{2} \log (\pi \nu \sigma^2) \right) - \frac{\nu + 1}{2} \sum_{i=1}^{K_1} \log \left( 1 + \frac{(x_i^+ - \mu)^2}{(1 - \lambda)^2 \sigma^2 \nu} \right) - \frac{\nu + 1}{2} \sum_{i=1}^{K_2} \log \left( 1 + \frac{(x_i^- - \mu)^2}{(1 - \lambda)^2 \sigma^2 \nu} \right)
\]

where \( x_i^+, i = 1, \cdots, K_1 \) are observations less than \( \mu \), \( x_i^-, i = 1, \cdots, K_2 \) are observations greater than \( \mu \). The parameters obtained from maximizing the likelihood function asymptotically converge to the true value. It is clear that the sample size determines the accuracy of the estimator. For estimation we use the minimization routine by Fletcher, Reeves, Polak, Ribiere (1992) to simultaneously estimate the four parameters. Because the algorithm is sensitive to the starting values, we always use multiple starting values.
4 Empirical Results

We obtain the MLE parameter estimates for the skew t-distribution for daily cross-sectional returns from Jan 1, 1991 to Dec 31, 2002. As discussed below, the results show that the skew t-distribution fits the data reasonably well.

The Chi-square goodness of fit test is used to assess overall performance. The histogram of Chi-square values over the 3027 days is shown in Figure 3(a). The goodness of fit test supports the proposed skew t-distribution for all 3027 days, since the 95% critical value of Chi-square test (with 50 bins and 4 estimated parameters) is 61.66. In the Figure 3(b) we present the empirical CDF of the cross sectional return and the CDF plot of the skew t-distribution with the estimated parameters on the day with the largest Chi-square value, hence the day in which there is maximum deviation between the actual and skew t-distribution. It shows that the proposed skew t-distribution provides a close fit for the daily cross sectional returns data.

Table 1 presents summary statistics for the parameter estimates. Over the 12 years, the market follows the skew t-distribution with the degrees of freedom averaging around 3.4. On average, the daily distribution has a negative location parameter and is slightly skewed to right. Also, the parameters have small estimation errors.

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Table 1: Summary statistics for the parameter estimators of the skew t distribution

The daily parameter estimates are presented in Figure 4. The analysis of the parameter
estimates shows that: (1) most of the degrees of freedom fall in the range of 2 to 5, which indicates that the cross sectional return follows the distribution with a fatter tail than normal; (2) the tail behavior over 12 years is very different from the behavior of the scale parameter (that is, unlike market variance, the tail did not increase during 1998); (3) the scale parameter estimate $\hat{\sigma}$ is very similar to the behavior of the market dispersion shown in section 2; (4) the sample measures presented in section 2 have high correlation with the corresponding model-based parameter estimates; (5) the degrees of freedom parameter estimates and the tail length index in section 2 have a significant linear relationship with correlation coefficient 0.80.

The cross sectional return tends to be positively skewed since the average of the skewness parameter is positive. This is different from the previous evidence that the distribution of stock returns is slightly negatively skewed. Among all 3027 skewness parameter estimates, 1067 of them are negative and 1960 of them are positive. When we examine the correlation
between the skewness parameter and the daily market index (S&P 500) return, we find a significant linear relationship. That is, when the market goes up (a positive return on average), the cross-sectional return distribution is positively skewed and when the market goes down (a negative return on average), it is negatively skewed. This fact is shown in Figure 5.

When we look at the inverse of the degrees of freedom estimates, an indicator of the tail-fatness of the underlying distribution, we detect a cyclic pattern of about four years. This cycle is different from the daily changes of the market volatility. The market volatility increases in 1998 and comes down, but tail-fatness shows no such tendency. On the other hand, the mean of the market return has similar change as the market dispersion. This is possibly due to high volatility being compensated with high return. Figure 6 shows that there is no clear trend in the ratio of $\mu/\sigma$. When the tail gets fatter, there are more stocks with extreme values. A portfolio manager who can catch the movement of the market tails
Figure 5: Scatter plot of skewness parameter and S&P 500 daily return, the dash line is least square reference line.

and pick the best and avoid the worst, will have increased profit. In this case, the changes in the tail-fatness correlate with profit opportunities.

5 Conclusion

In this paper we have presented values for the first four moments of the daily cross-sectional stock returns from 1991 to 2002. A skew t-distribution was proposed and fitted to the daily observations. Our proposed skew t-distribution provides a good fit to the data. The empirical moments closely follow the corresponding parameters in the skew t-distribution. We find the skewness parameter to be highly correlated with the overall market return. The dispersion of the cross section changed dramatically during the period while the tail index varied only slightly. The tail index value of about 3.4 indicates the fat-tailed nature of the cross-sectional distribution. Previous research had identified fat-tails with the return distribution of individual stocks.

The cross-sectional distribution provides a more comprehensive view of the market than a
univariate market average return. This is of particular concern to long-short equity managers whose return universe is the cross section of equity returns.

References


