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Robust voting

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Abstract. The formal equivalence between social choice and statistical estimation means that criteria used to evaluate estimators can be interpreted as features of voting rules. The robustness of an estimator means, in the context of social choice, insensitivity to departures from majority opinion. In this paper we consider the implications of substituting the median, a robust, high breakdown estimator, for Borda's mean. The robustness of the median makes the ranking method insensitive to outliers and reflect majority opinion. Among all methods that satisfy a majority condition, median ranks is the unique one that is monotonic. It is an attractive voting method when the goal is the collective assessment of the merits of alternatives.

1. Introduction

Ever since Kenneth Arrow put forth his impossibility theorem, economists, political scientists, and others interested in public choice have been exploring alternative approaches to group decision making. While various criteria have been investigated, little attention has been directed toward applying the statistical notion of robustness to voting problems. Robustness of an estimator is a measure of sensitivity to departures from an hypothesized model; see, for example, Koenker (1982). Related to robustness is the concept of the breakdown point, which measures an estimator's sensitivity to outlying observations. Translated to the voting context, the breakdown idea implies that the social ranking of alternatives depends on a large subset of voters without undue influence by a minority. This contrasts with methods such as the Borda count that rely on taking the mean, a notoriously nonrobust method. In this paper we consider the implications of substituting the median, a robust, high breakdown estimator, for Borda's mean.

The scant attention given to connections between voting and estimation problems is somewhat surprising in view of their formal equivalence; but see Levy (1989) for discussion of statistical ideas, robustness, and the problem of

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estimating the "center" of private interests. A ranking system maps vectors of voter rankings to an overall ranking for the group in the same way that a multivariate location estimator maps vectors of sample observations to an overall estimate. The equivalence between the problems suggests that criteria used to evaluate estimators can be reinterpreted for consideration as features of voting rules.

The high breakdown property of the median means that the ranking cannot be easily manipulated by a minority. In addition, and perhaps somewhat surprising, is the fact that median ranks is the only high breakdown method that also satisfies a monotonicity requirement. These properties are presented in the next section. Section 3 discusses the median in relation to other voting methods.

2. Properties of median ranks

Let there be a fixed menu of alternatives ($a_1, \dots a_S$) or (a,b,c, . . .), with each voter, i = 1, . . . ,n, ranking alternatives from best to worst. For simplicity suppose there are no ties. Individual rankings are denoted by $R_i = (r_{i1}, \dots, r_{iS})$ where r_{iS} is the rank assigned by the i^{th} voter to option s. A social decision function selects a single option – the winner – from the set of alternatives, while a social welfare function orders alternatives. In this paper we focus on the latter. Such a function takes the ranking vectors, $R_1, \dots, R_S$ into another ranking vector, $R_0 = (r_{01}, \dots, r_{0S})$ where r_{0s} is the overall rank of option s.

Jean-Charles de Borda, as early as 1770, advocated a simple summing of expressed voter preferences to achieve a social ranking. [See Young (1988) and McLean and Urken (1995) for discussions of the early history of voting schemes.] The Borda score gives zero points to a voter's least preferred option, 1 for the next option, and so on up to S points for the most preferred. The Borda ranking is then determined by ordering the Borda scores.

Borda's scheme received considerable support from LaPlace who justified it in terms of an equiprobability argument. But LaPlace was also among the first to notice that the Borda mechanism gives strong incentives for misrepresentation of preferences; see McLean and Urkin (1995) and Nanson ([1882],1907). LaPlace was particularly worried about situations in which the "best" alternative was relatively clear, but unscrupulous individuals might work to advance an alternative in their private interest. Where the Borda ranking is used and many alternatives are available, a determined minority can dramatically lower the chances of an otherwise popular choice.

To see the nature of the problem with the Borda count, consider the reported rankings shown in Table 1. There are voters, (1, . . . ,5) and alternatives, (a,b,c,d,e). The table shows that voter 1's reported ranking is $\langle$b,a,e,d,c$\rangle$

(so the ranking vector is $R_1 = (2,1,5,4,3)$). Notice that option "c" is ranked number "one" – best – by a majority of voters.

The next part of the table indicates the overall ranking based on the Borda criterion. The table shows that the Borda ranking vector is, $R_0 = (3,1,2,4,5)$ so the ordering of alternatives becomes $<b,c,a,d,e>$. Hence, "b" is top ranked even though a majority of voters have "c" as their preferred option. Such a result becomes particularly upsetting if we suspect that voters "1" and "2" misrepresented their rankings of option "c" in an effort to advance their most preferred option. From a statistical perspective, the problem with the Borda method (both in its failure to reflect the majority and in its susceptibility to manipulation) derives from its reliance on the average as an estimate of central tendency. That is, the Borda count is a linear function of an option's *average* rank; the average rank equals S (the number of alternatives) minus (n–1) times the Borda count. Hence, the Borda ranking is the same as the ranking that would be obtained from the vector whose components are the average ranks; the average ranks for the options in the example are shown in the last part of Table 1.

The statistical problem with the Borda count derives from its reliance on a nonrobust measure of location. The mean can be far from the bulk of the data; it is not robust. In a purely statistical context where the variables can take arbitrary numerical values, a single outlying observation causes the mean or average to depart from the bulk of the data. In statistics, this sensitivity to outlying observations is expressed by the concept of the breakdown point, which is, roughly, the largest proportion of outliers an estimate can tolerate without breaking; see, for example, Hampel et al. (1986). On this criterion, the statistical average could not be a worse estimator: it breaks with only one observation (one outlying observation can move the average an arbitrarily large distance). This contrasts with 50% breakdown methods, like the median, which remain near the majority of data even when $(50-\epsilon)\%$ of the data is outlying. In the voting context the rank-valued observations cannot take arbitrary values, thus limiting the extent to which a few voters can change the average rank. Still, an average statistic like the Borda count can be pulled from a majority opinion by a minority of voters.

The conviction that the overall ordering should reflect the majority, combined with the fact that the Borda method is the same as the average rank, suggests replacing Borda's average rank with a more robust estimate. One obvious possibility in this regard is to use the median rank. Instead of averaging the ranks, we take the median of the ranks. In the example this leads to the ordering, $<c,a,b,d?e>$; see the bottom of Table 1. The majority opinion that "c" is best implies that its median rank is equal to one and it is now the top-ranked alternative.[1]

Table 1.

Voter's rankings (best = 1)

Option	Voters 1	2	3	4	5
a	2	2	2	4	4
b	1	1	3	3	3
c	5	4	1	1	1
d	4	5	5	2	2
e	3	3	4	5	5

Borda matrix (n-rank)

Option	Voters 1	2	3	4	5	Borda count	
a	3	3	3	1	1	11	
b	4	4	2	2	2	14	Borda ranking
c	0	1	4	4	4	13	<b,c,a,d,e>
d	1	0	0	3	3	7	
e	2	2	1	0	0	5	

Condorcet ranking

Pairwise votes (row vs. col)

Option	a	b	c	d	e	Sum
a	–	1	2	3	5	11
b	4	–	2	3	5	14
c	3	3	–	4	3	13
d	2	2	1	–	2	7
e	0	0	2	3	–	5

Pairwise majority (1 = majority)

Option	a	b	c	d	e	
a	–	0	0	1	1	Condorcet ranking
b	1	–	0	1	1	<c,b,a,d,e>
c	1	1	–	1	1	
d	0	0	0	–	1	
e	0	0	0	0	–	

Statistics

Option	1	2	3	4	5	Ave	Majority	Median	
a	2	2	2	4	4	2.80	2	2	
b	1	1	3	3	3	2.20	3	3	Median ranking
c	5	4	1	1	1	2.40	1	1	<c,a,b,d?e>
d	4	5	5	2	2	3.60	?	4	
e	3	3	4	5	5	4.00	?	4	

The median is a 50% breakdown estimator, but it is not the only such statistic. Among other 50% breakdown estimators there is, for example, the least median of squares (LMS) or shortest half; see Rousseeuw (1984). The LMS is the midpoint of the half-subset of the observations that has the shortest range. For an option like "a" (or "b" or "c") where there is a majority subset of the data that has the same rank value, the shortest half has length zero and the LMS, median, and majority ranks are all equal. For an option like "d" where there is not a 50% cluster of identical rank values the shortest half is the interval [4,5] and the LMS is the midpoint 4.5. Notice that for option "e" there are multiple LMS estimates because there are two equivalent shortest halves [3,4] and [4,5] so that a tie-breaking rule would be needed to determine the final estimate.[2]

When there is a 50% cluster of identical rank values, the median, LMS, and – it turns out – all 50% breakdown estimators are equal; for a discussion of this exact-fit and other properties of 50% breakdown estimates; see Rousseeuw (1994) and Bassett (1991). In general, when there is not a 50% cluster of identical values, high breakdown estimates are determined by the narrowest "window width" of observations that encompasses 50% of the data.

The exact fit property of 50% breakdown methods means they all satisfy the following conception of majority rule.

> *Majority condition:* If option s has rank r_s for a majority of voters, and option s' has rank $r_{s'}$ for a majority of voters where $r_s < r_{s'}$, then $r_{0s} < r_{0s}$.

This condition guarantees that an option ranked best by a majority of voters (and there can be only one such option) will be ranked best overall. It goes on to generalize the idea of majority to positions other than first place. Thus if a majority of voters consider option "s" to be second best and a majority have option "s"' third best, then the condition requires that s be ranked better than "s"'. This conception of majority is discussed further below.

There is another condition satisfied by the median. The median cannot move in the opposite direction of a revised individual ranking; an improved individual ranking for an alternative cannot cause it to become worse in the social ordering. This monotonicity or non-negative responsiveness condition is as follows.

> *Monotonicity condition:* If an individual's ranking changes from R_i to R_i', which are identical except for options s and s′ where, $r_{is} > r_{is'}$, $r'_{is} < r'_{is'}$, and $r_{os} < r_{os'}$, then $r'_{os} < r'_{os'}$.

This condition is also satisfied by the Borda count. It generally fails for hierarchical voting methods such as those that use run-off contests. As the next example shows, it also does not hold for the LMS estimator.

Table 2.

Option	Voters' rankings 1	2	3	4	5	Statistics Ave	Maj	Med	LMS	Ranks Ave	Med	LMS
a	1	1	3	4	6	3.0	–	3	1.5	3	2 or 3	1
b	2	2	2	3	5	2.8	2.0	2	2	2	1	2
c	3	3	1	2	3	2.4	3.0	3	2.5	1	3 or 2	3
d	4	4	4	1	4	3.4	4.0	4	4	4	4	4
e	5	5	5	5	2	4.4	5.0	5	5	5	5	5
f	6	6	6	6	1	5.0	6.0	6	6	6	6	6

Option	Voters' rankings 1	2	3	4	5	Statistics Ave	Maj	Med	LMS	Ranks Ave	Med	LMS
a	1	1	3	4	4	2.6	–	3	3.5	2	2 or 3	3
b	2	2	2	3	5	2.8	2.0	2	2	3	1	1
c	3	3	1	2	3	2.4	3.0	3	3	1	3 or 2	2
d	4	4	4	1	6	3.8	4.0	4	4	4	4	4
e	5	5	5	5	2	4.4	5.0	5	5	5	5	5
f	6	6	6	6	1	5.0	6.0	6	6	6	6	6

Consider the rankings at the top of Table 2. The statistics show the LMS ranking is, <a,b,c,d,e,f>. Now consider how the rankings change in the second part of the table where the only difference is that voter 5 has "a" ranked fourth instead of sixth, and "d" ranked sixth instead of fourth. Notice what this has done to the ranking of "a" and "d". The improved ranking for "a" has caused it to become worse, to go from first to third. The LMS can thus respond negatively to a change in individual rankings.

If monotonicity and the majority condition are thought desirable then, it turns out, all voting systems are eliminated, except for median ranks. Systems other than the median that satisfy the majority condition do not satisfy monotonicity, and systems like Borda's that satisfy monotonicity do not satisfy the majority condition. This uniqueness result for the median method is developed in Bassett and Persky (1994).

3. Discussion

3.1. *Strategic voting-monotonicity*

Strategic voting occurs when a belief about how others will vote leads to a lowered ranking for a preferred candidate who is thought to have no chance against another, despised, candidate. An advantage of a robust system is that it implicitly sends a message to the voters or judges that strategic behavior will have a relatively small payoff. Robust methods are desirable when the behavior of voters, rather than predetermined, is instead partly influenced by the design of the voting mechanism. As LaPlace first noted, the degree of misrepresentation is at least partly an endogenous variable. Robustness reduces the incentive to act strategically.

For strategic voting to be possible it is necessary for voters to sense not only how others will vote, but also how their own vote will affect the outcome. If a system is not monotonic, it will not be clear how a change in ones individual ranking affects the overall outcome and hence how to behave strategically. For this reason it has been sometimes argued that the absence of monotonicity is actually a good thing; see the discussion in Levin and Nalebuff (1995).

A system that can move opposite expressed rankings may be sufficiently confusing to eliminate the incentive for strategic responses. But it can also lead to absent, lethargic, or random voting. Monotonicity might not be as critical for a statistical estimator where the data, lengths or weights, are without their own desires and need for justification. With voting, however, the data are individuals' opinions and have to be expressed; they cannot be read directly from the brain. Especially where voters are asked to engage in a serious and time-consuming assessment of alternatives, it is important to assure that votes will count in a positive direction. From this motivational perspective, then, it is important that the final ranking bear a nonnegative relation to voter rankings. Without a connection between expressed rankings and the final tally, voters might reasonably refrain from developing and expressing their views at all, or vote randomly, without regard to their true opinions.

While monotonicity leaves a system open to the problems of strategic voting, its promise of a correlation between individual and group rankings seems on balance a desirable component of a voting system. Median ranks meets the monotonicity condition. If one desires this and the majority rank condition, then there is nothing left except median ranks.

3.2. *Majority conditions*

The median condition for majority needs to be distinguished from the more familiar pairwise condition used by Condorcet. Condorcet was keenly aware of the shortcomings of the Borda system and advanced a different method for insuring majority decisiveness. The method is illustrated in the bottom portion of Table 1. The matrix shows the number of votes received by each option when paired against each competitor. This is followed by the matrix indicating the number of victories in all pairwise contests. Alternative "c" defeats all other options and hence, by the Condorcet criterion, it is ranked best. When an option is ranked best by a majority of voters – as is the case with "c" in

Table 3. Condorcet and median rank winners are not the same

Options	Voters 1	2	3	4	5	Med rank
a	1	1	3	3	3	3
b	2	2	2	4	2	2
c	5	3	4	1	5	4
d	4	4	5	2	1	4
e	3	5	1	5	4	4

Option “a” is the Condorcet winner: it wins all pairwise comparisons.
Option “b” is the median rank winner.

this example – it must defeat all other options in pairwise contests. Hence, both the Condorcet and median method guarantee that an option ranked best by a majority of voters will be at the top of the social ranking.

The Condorcet ordering of the remaining options proceeds sequentially by eliminating already-ranked alternatives. After excluding “c”, “b” wins all remaining pairwise contests, and hence is ranked second. In a similar fashion the Condorcet method leads to the ranking, <c,b,a,d,e>. As is well known (but not illustrated by this example) the Condorcet criterion has its own major problem, namely, it can cycle; “a” defeats “b”, “b” defeats “c”, and “c” defeats “a”. The method therefore does not necessarily produce an unambiguous ranking of alternatives.

To see how differences in the Condorcet and median-rank ideas of majority can occur, let a compact majority be one in which each voter in the majority possesses identical rankings over all alternatives. In this restricted situation the Condorcet and median approaches will necessarily agree (but possibly differ from the Borda count). Differences in rankings begin to arise as membership in the majorities changes. Consider options “a” and “b” in Table 1. For these alternatives, there are two distinct subsets of majority voters so that in the pairwise comparisons a majority of voters rank “b” better than “a” even though a majority have “b” in third and “a” in second place.

Differences between the Condorcet and median rank majority can also arise for the top-ranked alternative. Table 3 presents an example in which the Condorcet winner is not the median rank winner. Further, there can be a unique median rank winner even though the Condorcet method cycles. This is illustrated in Table 4.

There may well be, of course, collective assessments in which minority opinions provide valuable information. In such circumstances the decisiveness of a majority inherent in median rankings may not be desirable. But this hardly

Table 4. Unique median winner but no Condorcet winner

Options	Voters 1	2	3	4	5	Med rank
a	1	1	3	4	4	3
b	2	2	4	1	1	2
c	3	3	5	2	2	3
d	4	4	2	3	3	3
e	5	5	1	5	5	5

Option “a” defeats “b”, “c” and “e”. But “a” loses to “d”.
Option “b” is the median rank winner.

makes a strong argument for Condorcet’s method, which also emphasizes the majority. What is required when minority opinion is to be given greater weight is a move in the direction of a system like Borda’s.

3.3. *Independence of irrelevant alternatives*

Median ranks, like the Borda count, violates the independence of irrelevant alternatives condition; see McLean (1995). The desirability and application of median ranks therefore depends in part on the desirability of this well known condition for voting systems.

The independence requirement seems most desirable when the set of agenda options can be modified in response to a proposed voting method. Independence then eliminates the incentive to manipulate outcomes through a judicious selection of the choice set, or sequencing of the agenda. For this reason, the median is probably best suited to situations where there is a predetermined slate or the slate of alternatives is determined separately from the voting method.

Another reason sometimes given for the independence condition is that it serves as a check on the rationality of choice. On this view, the failure of independence is very serious. It would mean it was irrational to prefer “b” over “c” when the choice set was (b,c), but then prefer “c” over “b” when “a” was appended to the choice set..

This argument essentially makes binary comparisons the basis of rational choice. It parallels the logic of individual choice as developed by pairwise revealed preference. The inherent rationality of the binary approach and the independence condition, however, is arguable for individual and collective choice; see, e.g., Sen (1995a,1995b). Examples of menu-dependent preferences that violate independence do not seem irrational when the relative ranking of “b” and “c” reasonably does depend on other, potentially relevant, alternatives. Insistence on the independence of irrelevant alternatives would rule out situations in which one's preference ranking depends on the set of available options. Further, even if individual rankings respect binary comparisons, there seems no compelling reason for excluding menu-dependent collective judgements as irrational.

It should be noted that median ranks does possess a minimum sensitivity property. Removing (adding) one alternative from the menu can at most raise (lower) an option's median rank by one place. By way of contrast, the removal in a Borda ranking of a single option can change the ranking of another option by as many as n places, where n is the total number of voters.

3.4. *Committee rankings*

When considering alternative systems, it is useful to distinguish between different objectives. Economists typically think in terms of aggregating the individualistic preferences of consumers or voters. An input to this aggregation method represents a voter's evaluation of alternatives according to their own self interest. The design problem is then about how best to combine the given preferences/evaluations, without regard to how evaluations were formed or the opinions of other group members. This may be contrasted with (the typically nonmarket) situations in which individuals are asked to rank and compare alternatives from the point of view of the group as a whole. This alternative problem admits that individual evaluations might not be a priori given, and could be formed in light of group discussion about the relative merits of alternatives. From the group perspective there are individual self-interest issues that are supposed to be ignored when members develop their rankings. Still there may be concern that some members will base their rankings on criteria that are supposed to be irrelevant.

For example, in the international skating context, judges are supposed to evaluate performance without regard to skater nationality. This does not eliminate the possibility that judges might attempt to steer rankings to insure a higher placement for their own country. Or the setting may be a committee deciding on investment projects for a business, or priorities for a legislature. Evaluations are supposed to be on the merits of projects. The concern again is that the expressed rankings that are inputs to the aggregation method might reflect narrow self-interest (one's own pet projects, for example) rather than an honest evaluation of alternatives. It is not the subjectivity of the evaluation that causes the difficulty as all evaluations are subjective. The problem is that the self-interested evaluation differs from the blind evaluation that would occur if a skater's nationality could somehow be hidden.[3]

It was precisely such concerns that led to Condorcet's conception of voting as a collective search for the truth – an objective assessment of alternatives that would derive from subjective rankings. The problem was not seen as an aggregation of individuals with conflicting tastes, but as an exercise in finding the truth. In such circumstances, the robustness of median ranks protects against the influence of a few judges and may encourage discussion of alternatives from the group perspective. It sends a signal that self interested evaluations will not receive much weight in the final ranking.

4. Conclusion

Median ranks is the only robust method that satisfies the majority condition and responds nonnegatively to voter rankings. Its ranking can differ from the well-known methods associated with Borda and Condorcet. Because the method is not based on pairwise comparisons, it violates Arrow's independence condition. The method is therefore best suited to situations where there are predetermined alternatives. The robustness of the technique makes it relatively insensitive to outliers and to reflect majority opinion. It is attractive when individual rankings may involve "inappropriate" self-interest and where the goal is a collective assessment of the merits of alternatives.

Notes

1. The ordering of "d" and "e" with their identical medians is left undecided by the median-rank rule. When there are tied median ranks, the options can be declared indifferent, or the ties can be broken by applying additional rules; the "?" relation means that any of d>e, "d" indifferent to "e", or d<e could hold. An ordering is consistent with median ranks as long as options with better median ranks are ranked better in the overall ordering. Since median ranks can leave alternatives tied it requires that tie-breaking rules be appended to the method.
2. It should be noted that the median and other high breakdown methods are sensitive to data when there are two nearly equal-sized sets of observations or voters, which are both slightly smaller than a majority. In this case the high breakdown estimate seeking a majority of the data can shift with a change in only a small subset of the data; see Hettmansperger and Sheather (1992), Mosteller and Tukey (1977), and Levy (1989).
3. A similar concern with self-interested evaluations leads to consideration of a Rawlsian-like "original position" for evaluating income distributions. From the original position we can disinterestedly compare because we do not know our own place in the income distribution.

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